

Embedding Dimensions of Finite von Neumann Algebras

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Abstract: We introduce “embedding dimensions” of a family of generators of a finite von Neumann algebra when the von Neumann algebra can be faithfully embedded into the ultrapower of the hyperfinite II_1 factor. These embedding dimensions are von Neumann algebra invariants, i.e., do not depend on the choices of the generators. We also find values of these invariants for some specific von Neumann algebras.

1. Introduction

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H . A von Neumann algebra is a $*$ -subalgebra of $B(H)$ that is closed in the weak operator topology on $B(H)$.

In a series of remarkable papers, published between 1936 and 1943, Murray and von Neumann described some basic structures on von Neumann algebras. They separated the family of von Neumann algebras into three types, I, II, and III, and constructed examples for each type. In fact, Murray and von Neumann provided two methods for constructing type II_1 von Neumann algebras. One is obtained from the “left regular representation of a discrete infinite group.” The other is related to the action of such a group on a measure space (of finite measure) by measure preserving transformations.

The first construction proceeds as follows. Let H be $l^2(G)$. We assume that G is countable so that H is separable. For each g in G , let L_g be translation of functions in $l^2(G)$ by g^{-1} . Then $g \rightarrow L_g$ is a faithful unitary representation of G on H . Let $L(G)$ be the von Neumann algebra generated by $\{L_g : g \in G\}$. When each conjugacy class in G (other than that of the identity e) is infinite, $L(G)$ is a factor of type II_1 . In this case, we say that G is an infinite conjugacy class (i.c.c.) group.

Specific examples of such II_1 factors result from choosing for G any of the free groups F_n on n generators ($n \geq 2$), or the direct products of two free groups

$F_m \times F_p$, ($m, p \geq 2$), or the permutation group Π of the integers \mathbb{Z} (consisting of those permutations that leave fixed all but a finite subset of \mathbb{Z}). A factor is called “hyperfinite” if it is the ultraweak closure of the ascending union of a family of finite-dimensional self-adjoint subalgebras. A deep result of Murray and von Neumann shows that all such factors are isomorphic. Moreover, $L(\Pi)$ is that hyperfinite factor of type II_1 . Murray and von Neumann ([11]) show that $L(\Pi)$ is not $*$ -isomorphic to $L(F_n)$ ($n \geq 2$).

The second construction is more complicated. Let (X, μ) be a non-atomic measure space of finite measure. G (with unit e) is a countable (infinite) group of measure preserving transformations of X . Our Hilbert space is $L^2(X, \mu)$. Let \mathcal{A} be the commutative von Neumann algebra $L^\infty(X, \mu)$. Now G can be viewed as a group of automorphisms of the von Neumann algebra \mathcal{A} . Murray and von Neumann constructed a von Neumann algebra $R(\mathcal{A}, G)$ associated with the group G and the commutative von Neumann algebra \mathcal{A} . If G acts freely and ergodically on X , the von Neumann algebra $R(\mathcal{A}, G)$ is a type II_1 factor. In addition, \mathcal{A} is a maximal abelian subalgebra in $R(\mathcal{A}, G)$. The normalizers of \mathcal{A} (those unitary operators U in $R(\mathcal{A}, G)$ such that $U\mathcal{A}U^* = \mathcal{A}$) generate the von Neumann algebra $R(\mathcal{A}, G)$. We call a maximal abelian subalgebra of a finite von Neumann algebra, whose normalizers generate the full von Neumann algebra, a Cartan subalgebra.

It is a long standing open problem whether every II_1 von Neumann algebra has Cartan subalgebras. This question was answered by Voiculescu negatively after he introduced his remarkable theory of free entropy (see [17]), an analogue of classical entropy and Fisher information measure. Associated with the free entropy, he defined a free entropy dimension δ_0 which, in some sense, measures the “noncommutative dimension” of a space. In [17] he showed that for any n in \mathbb{N} , $\delta_0(L(F_n)) \geq n$. Soon he showed in [18] that if a von Neumann algebra \mathcal{N} has a Cartan subalgebra, then $\delta_0(\mathcal{N}) \leq 1$. Thus free group factors $L(F_n)$ ($n \geq 2$) have no Cartan subalgebra. Later, Ge ([5]) showed that if a von Neumann algebra \mathcal{N} is not prime, i.e., is a tensor product of two infinite-dimensional von Neumann algebras, then $\delta_0(\mathcal{N}) \leq 1$. In particular, $L(F_n) \not\cong L(F_m) \otimes L(F_p)$ for all $n, m, p \geq 2$.

In [10], we introduced upper free orbit dimension for finite von Neumann algebras, a concept closely related to Voiculescu’s free entropy dimension. By some easily obtained properties of upper free orbit dimension, we got very general results which imply most of the applications of Voiculescu’s free entropy dimension on finite von Neumann algebras.

It is well-known that Voiculescu’s free entropy dimension is closely related to Connes’ embedding problem which asks whether every separable type II_1 factor will be faithfully embedded into the ultrapower of the hyperfinite II_1 factor, \mathcal{R}^ω . In fact Voiculescu’s free entropy dimension of a finite von Neumann algebra can be viewed as a measurement of the number of ways to embed this von Neumann algebra into \mathcal{R}^ω . The upper free orbit-dimension of a von Neumann algebra, introduced in [10], can be view as a measurement of the number of ways to

embed, modulo conjugate actions by unitary elements of \mathcal{R}^ω , this von Neumann algebra into \mathcal{R}^ω .

In more details, suppose x_1, \dots, x_n is a family of generators of a finite von Neumann \mathcal{N} which can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$, where $\mathcal{M}_k(\mathbb{C})^\omega$ is the ultraproduct of $\{\mathcal{M}_k(\mathbb{C})\}_{k=1}^\infty$ along the free filter ω . Based on the philosophy in preceding paragraph, we define $\mathcal{H}_s^\omega(x_1, \dots, x_n)$, s -embedding dimension of x_1, \dots, x_n for $s \geq 0$, to be some measurement of the number of ways to embed \mathcal{N} into the ultrapower $\mathcal{M}_k(\mathbb{C})^\omega$. We show that \mathcal{H}_s^ω is a von Neumann algebra invariant, i.e. does not depend on the choices of families of generators. Then we carry out the computation of values of these invariants for finite von Neumann algebras. For example, if \mathcal{N} is a hyperfinite von Neumann algebra, then $\mathcal{H}_0^\omega(\mathcal{N}) = 0$. If \mathcal{N} is the free group factor on n generators, then $\mathcal{H}_1^\omega(\mathcal{N}) = \infty$. On the other hand, $\mathcal{H}_1^\omega(\mathcal{N}) = 0$ if \mathcal{N} is a type II_1 factor with Cartan subalgebras, a nonprime type II_1 factor, or some type II_1 factor with property T. Therefore, this invariant does give us information on the classification of type II_1 factors.

Because these “embedding dimensions” may have potential applications on Connes’ embedding problems, it is worthwhile to study them in more details. Having its motivations from Voiculescu’s free entropy dimension and from free orbit dimension of [10], we develop the theory of embedding dimensions from its own interest and keep the paper as self-contained as possible. Another motivation of the paper comes from the attempt to further classify II_1 von Neumann algebras whose Voiculescu’s free entropy dimensions are equal to 1, especially from the question whether the tensor products of free group factors have Cartan subalgebras. We can view $\mathcal{H}_s^\omega(\mathcal{N})$, s -embedding-dimension of finite von Neumann algebra \mathcal{N} , as an analogue of the classical fractal dimension in the subject of finite von Neumann algebras. It is not hard to see that $\mathcal{H}_s^\omega(\mathcal{N})$ is a decreasing function of $s \geq 0$. We know that $\mathcal{H}_1^\omega(\mathcal{N}) = 0$ for many type II_1 factors (for example, type II_1 factors with Cartan subalgebras). Hopefully $\mathcal{H}_s^\omega(\mathcal{N})$ becomes non zero when s is small enough. In this direction we wish that these “fractal” dimensions can provide us with new tools to further classify type II_1 factors whose Voiculescu’s free entropy dimensions are equal to 1.

The organization of the paper is as follows. In section 2, we introduce some notations and give the definitions of embedding dimensions. We show that these embedding dimensions are von Neumann algebra invariants in section 3. The embedding dimensions of abelian von Neumann algebras and free group factors are obtained in section 4. Some technical lemmas on covering numbers and unitary orbit covering numbers are proved in section 5. The computation of values of embedding dimension for some specific type II_1 factors is carried out in section 6.

2. Some notations and definitions

2.1. Covering numbers. Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with entries in \mathbb{C} , and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$, i.e., $\tau_k = \frac{1}{k} \text{Tr}$, where Tr is the usual trace on $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{U}(k)$ denote the group of all unitary matrices in $\mathcal{M}_k(\mathbb{C})$. Let $\mathcal{M}_k(\mathbb{C})^n$ denote the direct sum of n copies of $\mathcal{M}_k(\mathbb{C})$. Let $\|\cdot\|_2$ denote the trace norm induced by τ_k on $\mathcal{M}_k(\mathbb{C})^n$, i.e.,

$$\|(A_1, \dots, A_n)\|_2^2 = \tau_k(A_1^* A_1) + \dots + \tau_k(A_n^* A_n)$$

for all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$.

DEFINITION 1. For every $\delta > 0$, we define the δ -ball $\text{Ball}(B_1, \dots, B_n; \delta)$ centered at (B_1, \dots, B_n) in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$ such that $\|(A_1, \dots, A_n) - (B_1, \dots, B_n)\|_2 < \delta$.

DEFINITION 2. For every $\delta > 0$, we define the δ -orbit-ball $\mathcal{U}(B_1, \dots, B_n; \delta)$ centered at (B_1, \dots, B_n) in $\mathcal{M}_k(\mathbb{C})^n$ to be the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all (A_1, \dots, A_n) in $\mathcal{M}_k(\mathbb{C})^n$ such that there exists some unitary matrix W in $\mathcal{U}(k)$ satisfying

$$\|(A_1, \dots, A_n) - (W B_1 W^*, \dots, W B_n W^*)\|_2 < \delta.$$

DEFINITION 3. For every $R > 0$, denote by $(\mathcal{M}_k(\mathbb{C})^n)_R$ the subset of $\mathcal{M}_k(\mathbb{C})^n$ consisting of all these (A_1, \dots, A_n) satisfying $\max_{1 \leq j \leq n} \|A_j\| \leq R$.

DEFINITION 4. Let Γ be a subset of $\mathcal{M}_k(\mathbb{C})^n$. (i) For $\delta > 0$, we define the δ -covering number $\nu_2(\Gamma, \delta)$ to be the minimal number of δ -balls that cover Γ with the centers of these δ -balls in Γ . (ii) Define the δ -orbit covering number $\nu(\Gamma, \delta)$ to be the minimal number of δ -orbit-balls that cover Γ with the centers of these δ -orbit-balls in Γ .

2.2. Embedding dimensions. Let $\mathcal{M}_k(\mathbb{C})$ be the $k \times k$ full matrix algebra with complex entries and τ_k be the normalized trace on $\mathcal{M}_k(\mathbb{C})$. If ω is a free filter in $\beta(\mathbb{N}) \setminus \mathbb{N}$ then denote by $\mathcal{M}_k(\mathbb{C})^\omega$ the quotient of the von Neumann algebra $l^\infty(\mathbb{N}, \prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C}))$ by the 0-ideal of the trace τ_ω , where τ_ω is defined by $\tau_\omega((X_k)_k) = \lim_{k \rightarrow \omega} \tau_k(X_k)$ for every $x = (X_k)_k$ in $\mathcal{M}_k(\mathbb{C})^\omega$. We also define the Hilbert norm $\|\cdot\|_{2,\omega}$ on $\mathcal{M}_k(\mathbb{C})^\omega$ by

$$\|x\|_{2,\omega} = \|(X_k)_k\|_{2,\omega} = \tau_\omega([(X_k)_k]^* [(X_k)_k])^{1/2} = \lim_{k \rightarrow \omega} \|X_k\|_2$$

for all $x = (X_k)_k$ in $\mathcal{M}_k(\mathbb{C})^\omega$. Then $\mathcal{M}_k(\mathbb{C})^\omega$ is a type II_1 factor; and τ_ω is a tracial trace on $\mathcal{M}_k(\mathbb{C})^\omega$.

Let \mathcal{N} be a finitely generated von Neumann algebra with a tracial state τ . Assume that \mathcal{N} can be faithfully embedded into $\mathcal{M}_k(\mathbb{C})^\omega$.

DEFINITION 5. We have the following definitions.

(i) Define $\Theta(\mathcal{N}, \mathcal{M}_k(\mathbb{C})^\omega)$ as a subset of $\text{Hom}(\mathcal{N}, \mathcal{M}_k(\mathbb{C})^\omega)$ consisting of all faithful trace-preserving embedding θ from (\mathcal{N}, τ) into $(\mathcal{M}_k(\mathbb{C})^\omega, \tau_\omega)$.

(ii) Define Ξ to be the set consisting of all sequences, $\xi = \{k_m\}_{m=1}^\infty$, of positive integers such that $\lim_{m \rightarrow \infty} k_m = \infty$.

(iii) We will also introduce “Voiculescu’s topological structure” on the space

$$\mathfrak{T} = \left(\prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C}) \right)^n = \prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C}) \times \cdots \times \prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C}).$$

First we introduce the neighborhood in \mathfrak{T} , indexed by elements y_1, \dots, y_n in $\mathcal{M}_k(\mathbb{C})^\omega$, $R > 0$ and $\xi = \{k_m\}_{m=1}^\infty$ in Ξ as follows. Define the neighborhood

$$\mathfrak{N}_{R,\xi}(y_1, \dots, y_n)$$

as a subset of $\prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C}) \times \cdots \times \prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C})$ consisting of all

$$((Y_{1,k})_{k=1}^\infty, \dots, (Y_{n,k})_{k=1}^\infty)$$

in $\prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C}) \times \cdots \times \prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C})$ such that $\|Y_{i,k}\| < R$ for all $1 \leq i \leq n, k \geq 1$ and

$$|\tau_k(Y_{j_1}^{\epsilon_1} \cdots Y_{j_p}^{\epsilon_p}) - \tau_\omega(y_{j_1}^{\epsilon_1} \cdots y_{j_p}^{\epsilon_p})| < \frac{1}{m},$$

for all $k \geq k_m$, $1 \leq p \leq m$, $1 \leq j_1, \dots, j_p \leq n$ and $\epsilon_i \in \{*, 1\}$ for $1 \leq i \leq p$.

(iv) For each $k \geq 1$, let \mathbb{P}_k be the projection from $(\prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C}))^n$ onto $(\mathcal{M}_k(\mathbb{C}))^n$.

REMARK 1. The introductions of the set Ξ and the “topology” are very necessary. Because the first finitely many terms of any representative of an element in $\mathcal{M}_k(\mathbb{C})^\omega$ can be chosen arbitrarily, we use this Ξ to exclude this “arbitrary” phenomena. Each ξ in Ξ plays the role of radius. And it is not hard to see that

$$\bigcup_{\{y_1, \dots, y_n\} \subset \mathcal{M}_k(\mathbb{C})^\omega} \left(\bigcup_{R>0, \xi \in \Xi} \mathfrak{N}_{R,\xi}(y_1, \dots, y_n) \right) = l^\infty(\mathbb{N}, \prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C}) \times \cdots \times \prod_{k=1}^\infty \mathcal{M}_k(\mathbb{C})).$$

DEFINITION 6. Suppose that x_1, \dots, x_n is a family of elements in \mathcal{N} . Let R be a positive number and $\xi = \{k_m\}_{m=1}^\infty$ be in Ξ . We define the neighborhood in \mathfrak{T} , indexed by x_1, \dots, x_n in \mathcal{N} , $R > 0$ and $\xi \in \Xi$, by

$$\mathfrak{N}_{R,\xi}(x_1, \dots, x_n) = \bigcup_{\theta \in \Theta(\mathcal{N}, \mathcal{M}_k(\mathbb{C})^\omega)} \mathfrak{N}_{R,\xi}(\theta(x_1), \dots, \theta(x_n)).$$

Voiculescu’s embedding dimension of x_1, \dots, x_n , $\delta_0^\omega(x_1, \dots, x_n)$, is defined by

$$\begin{aligned} \delta_0^\omega(x_1, \dots, x_n; R, \xi, \delta) &= \lim_{k \rightarrow \omega} \frac{\log(\nu_2(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n)), \delta))}{-k^2 \log \delta} \\ \delta_0^\omega(x_1, \dots, x_n) &= \limsup_{\delta \rightarrow 0} \sup_{R>0, \xi \in \Xi} \delta_0^\omega(x_1, \dots, x_n; R, \xi, \delta) \end{aligned}$$

For $s \geq 0$, we define s -embedding dimension of x_1, \dots, x_n , $\mathcal{H}_s^\omega(x_1, \dots, x_n)$, by

$$\begin{aligned}\mathcal{H}_s^\omega(x_1, \dots, x_n; R, \xi, \delta) &= \lim_{k \rightarrow \omega} \frac{\log(\nu(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n)), \delta))}{k^{2s}} \\ \mathcal{H}_s^\omega(x_1, \dots, x_n) &= \limsup_{\delta \rightarrow 0} \sup_{R > 0, \xi \in \Xi} \mathcal{H}_s^\omega(x_1, \dots, x_n; R, \xi, \delta)\end{aligned}$$

Generally, given any function $f(s, \cdot)$ where s is a parameter or a family of parameters, we define $f(s, \cdot)$ -dimension of x_1, \dots, x_n , $\mathcal{H}_{f(s, \cdot)}^\omega(x_1, \dots, x_n)$, by

$$\begin{aligned}\mathcal{H}_{f(s, \cdot)}^\omega(x_1, \dots, x_n; R, \xi, \delta) &= \lim_{k \rightarrow \omega} \frac{\log(\nu(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n)), \delta))}{f(s, k)} \\ \mathcal{H}_{f(s, \cdot)}^\omega(x_1, \dots, x_n) &= \limsup_{\delta \rightarrow 0} \sup_{R > 0, \xi \in \Xi} \mathcal{H}_{f(s, \cdot)}^\omega(x_1, \dots, x_n; R, \xi, \delta)\end{aligned}$$

REMARK 2. It follows from the definition that $\mathcal{H}_s^\omega(x_1, \dots, x_n; R, \xi, \delta)$ is a decreasing function of $\delta > 0$ and $\mathcal{H}_s^\omega(x_1, \dots, x_n)$ is a decreasing function of $s \geq 0$.

REMARK 3. When $s > 1$, we have $\mathcal{H}_s^\omega(x_1, \dots, x_n) = 0$ for all x_1, \dots, x_n in \mathcal{N} .

REMARK 4. Voiculescu's embedding dimension could be viewed as a measurement of the number of the ways to embed \mathcal{N} into $\mathcal{M}_k(\mathbb{C})^\omega$. For every embedding θ from \mathcal{N} into $\mathcal{M}_k(\mathbb{C})^\omega$, we know that $u\theta(\cdot)u^*$ is also an embedding from \mathcal{N} into $\mathcal{M}_k(\mathbb{C})^\omega$, where u is a unitary element of $\mathcal{M}_k(\mathbb{C})^\omega$. Therefore the “ s -embedding-dimension”, or \mathcal{H}_s , could be viewed as a measurement of the number of the ways to embed \mathcal{N} , modulo conjugate actions by unitary elements of $\mathcal{M}_k(\mathbb{C})^\omega$, into $\mathcal{M}_k(\mathbb{C})^\omega$.

We should also define the embedding dimensions of a family of elements x_1, \dots, x_n in the presence of another family of elements y_1, \dots, y_p of \mathcal{N} .

DEFINITION 7. Suppose $x_1, \dots, x_n, y_1, \dots, y_p$ are the elements of \mathcal{N} . Let $R > 0$, ξ be in Ξ and θ be in $\Theta(\mathcal{N}, \mathcal{M}_k(\mathbb{C})^\omega)$. Let

$$\mathfrak{N}_{R,\xi}(\theta(x_1), \dots, \theta(x_n) : \theta(y_1), \dots, \theta(y_p))$$

be the image of the projection of $\mathfrak{N}_{R,\xi}(\theta(x_1), \dots, \theta(x_n), \theta(y_1), \dots, \theta(y_p))$ onto the first n components, i.e.,

$$((A_{1,k})_k, \dots, (A_{n,k})_k) \in \mathfrak{N}_{R,\xi}(\theta(x_1), \dots, \theta(x_n) : \theta(y_1), \dots, \theta(y_p))$$

if there are elements $B_{1,k}, \dots, B_{p,k}$ in $\mathcal{M}_k(\mathbb{C})$ such that

$$((A_{1,k})_k, \dots, (A_{n,k})_k, (B_{1,k})_k, \dots, (B_{p,k})_k) \in \mathfrak{N}_{R,\xi}(\theta(x_1), \dots, \theta(x_n), \theta(y_1), \dots, \theta(y_p)).$$

Then we define,

$$\begin{aligned}
& \mathfrak{N}_{R,\xi}(x_1, \dots, x_n : y_1, \dots, y_p) \\
&= \bigcup_{\theta \in \Theta(\mathcal{N}, \mathcal{M}_k(\mathbb{C})^\omega)} \mathfrak{N}_{R,\xi}(\theta(x_1), \dots, \theta(x_n) : \theta(y_1), \dots, \theta(y_p)), \\
& \delta_0^\omega(x_1, \dots, x_n : y_1, \dots, y_p; R, \xi, \delta) \\
&= \lim_{k \rightarrow \omega} \frac{\log(\nu_2(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n : y_1, \dots, y_p)), \delta))}{-k^2 \log \delta} \\
& \delta_0^\omega(x_1, \dots, x_n : y_1, \dots, y_p) \\
&= \limsup_{\delta \rightarrow 0} \sup_{R > 0, \xi \in \Xi} \delta_0^\omega(x_1, \dots, x_n : y_1, \dots, y_p; R, \xi, \delta)
\end{aligned}$$

And

$$\mathcal{H}_s^\omega(x_1, \dots, x_n : y_1, \dots, y_p), \quad \mathcal{H}_{f(s, \cdot)}^\omega(x_1, \dots, x_n : y_1, \dots, y_p)$$

are defined similarly.

3. \mathcal{H}_s^ω is a von Neumann algebra invariant

In this section, we are going to show that \mathcal{H}_s^ω is a von Neumann algebra invariant, i.e. it does not depend on the choices of the generators. First, we have the following lemma which follows directly from the definition of embedding dimensions.

LEMMA 1. Suppose \mathcal{N} is a finitely generated von Neumann algebra with a tracial state τ and \mathcal{N} can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$. Let $x_1, \dots, x_n, y_1, \dots, y_p$ be elements in a von Neumann algebra \mathcal{N} . If x_1, \dots, x_n generate \mathcal{N} as a von Neumann algebra, then, for every $s \geq 0$,

$$\mathcal{H}_s^\omega(x_1, \dots, x_n) = \mathcal{H}_s^\omega(x_1, \dots, x_n : y_1, \dots, y_p).$$

PROOF. Note that x_1, \dots, x_n generate \mathcal{N} and y_1, \dots, y_p are contained in \mathcal{N} . Thus y_1, \dots, y_p can be approximated by the polynomials of x_1, \dots, x_n in $\|\cdot\|_2$ -norm. Let $R > \max\{\|x_i\|, \|y_j\|, 1 \leq i \leq n, 1 \leq j \leq p\}$ and $\xi = \{k_m\}_{m=1}^\infty$ be in Ξ . For each $m \geq 0$, there is $m' \geq 0$ satisfying: for every

$$\{X_1, \dots, X_n, Y_1, \dots, Y_p\} \subset (M_k(\mathbb{C}))_R,$$

if

$$|\tau_k(X_{j_1}^{\epsilon_1} \dots X_{j_q}^{\epsilon_q}) - \tau(x_{j_1}^{\epsilon_1} \dots x_{j_q}^{\epsilon_q})| < \frac{1}{m'},$$

for all $\{X_{j_i}\}_{i=1}^q \subset \{X_1, \dots, X_n\}$, $\{\epsilon_i\}_{i=1}^q \subset \{*, 1\}$ and $1 \leq q \leq m'$, then

$$|\tau_k(Z_{j_1}^{\epsilon_1} \dots Z_{j_q}^{\epsilon_q}) - \tau(z_{j_1}^{\epsilon_1} \dots z_{j_q}^{\epsilon_q})| < \frac{1}{m},$$

for all $\{Z_{j_i}\}_{i=1}^q \subset \{X_1, \dots, X_n, Y_1, \dots, Y_p\}$, $\{z_{j_i}\}_{i=1}^q \subset \{x_1, \dots, x_n, y_1, \dots, y_p\}$, $\{\epsilon_i\}_{i=1}^q \subset \{*, 1\}$ and $1 \leq q \leq m$.

Let $\tilde{\xi} = \{\tilde{k}_{m'}\}_{m'=1}^{\infty}$ in Ξ such that $\tilde{k}_{m'} = k_m$. Then

$$\mathfrak{N}_{R,\tilde{\xi}}(x_1, \dots, x_n) \subset \mathfrak{N}_{R,\xi}(x_1, \dots, x_n : y_1, \dots, y_p) \subset \mathfrak{N}_{R,\xi}(x_1, \dots, x_n),$$

for all $k \geq 1$. The rest follows from the definitions. \square

Now we are ready to show the main result in this section.

THEOREM 1. *Suppose \mathcal{N} is a von Neumann algebra with a tracial state τ and \mathcal{N} can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$. Suppose $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_p\}$ are two families of generators of \mathcal{N} . Then, for all $s \geq 0$,*

$$\mathcal{H}_s^\omega(x_1, \dots, x_n) = \mathcal{H}_s^\omega(y_1, \dots, y_p).$$

PROOF. We need only to show that

$$\mathcal{H}_s^\omega(x_1, \dots, x_n) \geq \mathcal{H}_s^\omega(y_1, \dots, y_p).$$

Since x_1, \dots, x_n are elements in \mathcal{N} that generate \mathcal{N} as a von Neumann algebra, for every $0 < \delta < 1$, there exists a family of noncommutative polynomials $\psi_i(x_1, \dots, x_n)$, $1 \leq i \leq p$, such that

$$\sum_{i=1}^p \|y_i - \psi_i(x_1, \dots, x_n)\|_2^2 < \left(\frac{\delta}{4}\right)^2.$$

Therefore, for every θ in $\Theta(\mathcal{N}, \mathcal{M}_k(\mathbb{C})^\omega)$, we have

$$\sum_{i=1}^p \|\theta(y_i) - \psi_i(\theta(x_1), \dots, \theta(x_n))\|_2^2 < \left(\frac{\delta}{4}\right)^2.$$

For such a family of polynomials ψ_1, \dots, ψ_p , and every $R > 0$ there always exists a constant $D \geq 1$, depending only on R, ψ_1, \dots, ψ_p , such that

$$\left(\sum_{i=1}^p \|\psi_i(A_1, \dots, A_n) - \psi_i(B_1, \dots, B_n)\|_2^2 \right)^{1/2} \leq D \|(A_1, \dots, A_n) - (B_1, \dots, B_n)\|_2,$$

for all $(A_1, \dots, A_n), (B_1, \dots, B_n)$ in $\mathcal{M}_k(\mathbb{C})^n$, all $k \in \mathbb{N}$, satisfying $\|A_j\|, \|B_j\| \leq R$, for $1 \leq j \leq n$.

For $R > 1, \xi \in \Xi$, by the definition of $\mathfrak{N}_{R,\xi}(\theta(y_1), \dots, \theta(y_p), \theta(x_1), \dots, \theta(x_n))$, there is some m_0 (only depending on δ , not on θ) such that, when $k \geq k_{m_0}$, every

$$(\tilde{H}_1, \dots, \tilde{H}_p, \tilde{A}_1, \dots, \tilde{A}_n) \in \mathbb{P}_k(\mathfrak{N}_{R,\xi}(\theta(y_1), \dots, \theta(y_p), \theta(x_1), \dots, \theta(x_n)))$$

satisfies

$$\left| \left(\sum_{i=1}^p \|\tilde{H}_i - \psi_i(\tilde{A}_1, \dots, \tilde{A}_n)\|_2^2 \right)^{1/2} - \left(\sum_{i=1}^p \|\theta(y_i) - \psi_i(\theta(x_1), \dots, \theta(x_n))\|_2^2 \right)^{1/2} \right| \leq \frac{\delta}{4}$$

Hence,

$$\left(\sum_{i=1}^p \|\tilde{H}_i - \psi_i(\tilde{A}_1, \dots, \tilde{A}_n)\|_2^2 \right)^{1/2} \leq \frac{\delta}{2}.$$

It follows that, when $k \geq k_{m_0}$, every

$$(H_1, \dots, H_p, A_1, \dots, A_n) \in \mathbb{P}_k(\mathfrak{N}_{R,\xi}(y_1, \dots, y_p, x_1, \dots, x_n))$$

satisfies

$$\left(\sum_{i=1}^p \|H_i - \psi_i(A_1, \dots, A_n)\|_2^2 \right)^{1/2} \leq \frac{\delta}{2}.$$

It is obvious that such an (A_1, \dots, A_n) is also in $\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n : y_1, \dots, y_p))$, which is contained in $\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n))$. On the other hand, by the definition of the orbit covering number, we know there exists a set $\{\mathcal{U}(B_1^\lambda, \dots, B_n^\lambda, \frac{\delta}{4D})\}_{\lambda \in \Lambda_k}$ of $\frac{\delta}{4D}$ -orbit-balls that cover $\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n))$ with the cardinality of Λ_k satisfying $|\Lambda_k| = \nu(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n)), \frac{\delta}{4D})$. Thus for such (A_1, \dots, A_n) in $\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n))$, there exists some $\lambda \in \Lambda_k$, such that

$$(A_1, \dots, A_n) \in \mathcal{U}(B_1^\lambda, \dots, B_n^\lambda, \frac{\delta}{4D}),$$

i.e., there is some $W \in \mathcal{U}(k)$ such that

$$\|(A_1, \dots, A_n) - (WB_1^\lambda W^*, \dots, WB_n^\lambda W^*)\|_2 \leq \frac{\delta}{4D}.$$

It follows that

$$\sum_{i=1}^p \|H_i - W\psi_i(B_1^\lambda, \dots, B_n^\lambda)W^*\|_2^2 = \sum_{i=1}^p \|H_i - \psi_i(WB_1^\lambda W^*, \dots, WB_n^\lambda W^*)\|_2^2 \leq \delta^2,$$

for some $\lambda \in \Lambda_k$ and $W \in \mathcal{U}(k)$, i.e.,

$$(H_1, \dots, H_p) \in \mathcal{U}(\psi_1(B_1^\lambda, \dots, B_n^\lambda), \dots, \psi_p(B_1^\lambda, \dots, B_n^\lambda); \delta).$$

Hence, by the definition of embedding dimension, we get

$$\begin{aligned}
0 \leq \mathcal{H}_s^\omega(y_1, \dots, y_p : x_1, \dots, x_n; R, \xi, 2\delta) &\leq \lim_{k \rightarrow \omega} \frac{\log(|\Lambda_k|)}{k^{2s}} \\
&= \lim_{k \rightarrow \omega} \frac{\log(\nu(\mathbb{P}(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n)), \frac{\delta}{4D}))}{k^{2s}} \\
&\leq \mathcal{H}_s^\omega(x_1, \dots, x_n),
\end{aligned}$$

since

$$\mathcal{H}_s^\omega(x_1, \dots, x_n; R, \xi, \delta_1) = \sup_{\delta_1 > 0} \sup_{R > 0, \xi \in \Xi} \lim_{k \rightarrow \omega} \frac{\log(\nu(\mathbb{P}(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n)), \delta_1))}{k^{2s}}.$$

Therefore $\mathcal{H}_s^\omega(y_1, \dots, y_p : x_1, \dots, x_n) \leq \mathcal{H}_s^\omega(x_1, \dots, x_n)$. Now it follows from Lemma 1 that

$$\mathcal{H}_s^\omega(y_1, \dots, y_p) = \mathcal{H}_s^\omega(y_1, \dots, y_p : x_1, \dots, x_n).$$

Hence $\mathcal{H}_s^\omega(y_1, \dots, y_p) \leq \mathcal{H}_s^\omega(x_1, \dots, x_n)$, which completes the proof. \square

Because of the preceding theorem, the following definition is well-defined.

DEFINITION 8. *Suppose \mathcal{N} is a finitely generated von Neumann algebra with a tracial state τ and \mathcal{N} can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$ for some free filter ω in $\beta(\mathbb{N}) \setminus \mathbb{N}$. Then, we define the s -embedding-dimension of \mathcal{N} by*

$$\mathcal{H}_s^\omega(\mathcal{N}) = \mathcal{H}_s^\omega(x_1, \dots, x_n),$$

for any family of generator x_1, \dots, x_n of \mathcal{N} .

REMARK 5. *It is trivial to see that $\mathcal{H}_s^\omega(\mathcal{N})$ is a decreasing function of $s \geq 0$ for every finite von Neumann algebra \mathcal{N} . And $\mathcal{H}_s^\omega(\mathcal{N}) = 0$ for all $s > 1$.*

REMARK 6. *We did not prove the Voiculescu's embedding dimension is a von Neumann algebra invariant.*

4. Embedding dimensions of abelian von Neumann algebras and free group factors

In this section, we are going to compute values of the embedding dimensions of abelian von Neumann algebras and free group factors.

The following lemma was first proved by Voiculescu in [17]. A simplified proof can be found in [2]. For the sake of completeness, we also sketch its proof here.

LEMMA 2. *Let x be a self-adjoint element in a von Neumann algebra \mathcal{N} with a tracial state τ . Let $R > \|x\|$. For every $\delta > 0$, there is some positive integer m such that, for all $k \geq 1$, if A, B are two self-adjoint matrices in $\mathcal{M}_k(\mathbb{C})$ satisfying $\|A\| \leq R, \|B\| \leq R$ and*

$$|\tau_k(A^p) - \tau(x^p)| < \frac{1}{m}; \quad |\tau_k(B^p) - \tau(x^p)| < \frac{1}{m},$$

for all $1 \leq p \leq m$, then there is some unitary matrix U in $\mathcal{U}(k)$ such that

$$\|UAU^* - B\|_2 \leq \delta.$$

PROOF. Suppose on the contrary that the following holds: there is some $\delta_0 > 0$ such that for every $m \geq 1$, there is some $k_m \geq 1$ and some self-adjoint matrices A_m, B_m in $\mathcal{M}_{k_m}(\mathbb{C})$ satisfying $\|A_m\| \leq R, \|B_m\| \leq R$,

$$|\tau_{k_m}(A_m^p) - \tau(x^p)| < \frac{1}{m}; \quad |\tau_{k_m}(B_m^p) - \tau(x^p)| < \frac{1}{m},$$

for all $1 \leq p \leq m$, and $\|UA_mU^* - B_m\|_2 > \delta_0$ for all unitary matrix U in $\mathcal{U}(k_m)$.

Let ω be a free filter in $\beta(\mathbb{N}) \setminus \mathbb{N}$. Denote by $\mathcal{M}_{k_m}(\mathbb{C})^\omega$ the ultrapower of $\{\mathcal{M}_{k_m}(\mathbb{C})\}_{m=1}^\infty$ along the filter ω . It is not hard to see $[(A_m)_m], [(B_m)_m]$ are two self-adjoint elements in $\mathcal{M}_{k_m}(\mathbb{C})^\omega$ that have the same distribution. By Lemma 7.1 of [13], there is some u in $\mathcal{M}_{k_m}(\mathbb{C})^\omega$, such that $u[(A_m)_m]u^* = [(B_m)_m]$. Let $(U_m)_m$ be a representative of u in $\mathcal{M}_{k_m}(\mathbb{C})^\omega$. We can assume that each U_m is a unitary matrix in $\mathcal{M}_{k_m}(\mathbb{C})$. Hence $\lim_{m \rightarrow \omega} \|U_m A_m U_m^* - B_m\|_2 = 0$, which contradicts with the assumption that $\|UA_mU^* - B_m\|_2 > \delta_0$ for all unitary matrix U in $\mathcal{U}(k_m)$. Therefore, the statement of the lemma is true. \square

REMARK 7. *The proof of the preceding lemma shows the same statement also holds for a unitary element x in \mathcal{N} (considering $*$ -distribution of a unitary element instead of distribution of a self-adjoint element). In fact, a stronger result was obtained in [2] in the case when x_1, \dots, x_n generate a hyphenfinite von Neumann algebra.*

THEOREM 2. *Suppose \mathcal{A} is an abelian von Neumann algebra with a tracial state τ . Then $\mathcal{H}_0^\omega(\mathcal{A}) = 0$.*

PROOF. By [11], we can assume that the abelian von Neumann algebra \mathcal{A} is generated by a self-adjoint element x . It is well-known that every abelian von Neumann algebra with a tracial state can be faithfully trace-preserving embedded into the ultrapower $\mathcal{M}_k(\mathbb{C})^\omega$. Let $\delta > 0, R > 0$ and $\xi = \{k_m\}_{m=1}^\infty$ be in Ξ . For every

$$A_k, B_k \in \mathbb{P}_k(\mathfrak{N}_{R,\xi}(x)),$$

there are some θ_1, θ_2 in $\Theta(\mathcal{A}, \mathcal{M}_k(\mathbb{C})^\omega)$ such that

$$A_k = \mathbb{P}_k(\mathfrak{N}_{R,\xi}(\theta_1(x))); \quad B_k = \mathbb{P}_k(\mathfrak{N}_{R,\xi}(\theta_2(x))).$$

Or

$$|\tau_k(A^p) - \tau_\omega(\theta_1(x))| < \frac{1}{m}; \quad |\tau_k(B^p) - \tau_\omega(\theta_2(x))| < \frac{1}{m}$$

for all $1 \leq p \leq m$ and $k \geq k_m$. From Lemma 2, it follows that when k is big enough there is some U_k in $\mathcal{U}(k)$ such that

$$\|U_k A_k U_k^* - B_k\|_2 \leq \delta.$$

This implies that the δ -orbit-covering number $\nu(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x)), \delta) = 1$. Therefore, $\mathcal{H}_0^\omega(x) = 0$. By Theorem 1, we obtain that $\mathcal{H}_0^\omega(\mathcal{A}) = 0$. \square

REMARK 8. *By the remark after Lemma 2, the following result also holds. If \mathcal{N} is a hyperfinite von Neumann algebra with a tracial state, then $\mathcal{H}_0^\omega(\mathcal{N}) = 0$.*

The following proposition is Theorem 2.7 of [16], whose proof depends on the powerful tools from random matrices. An elementary proof which is based on the basic facts of unitary matrices can be found in [2].

PROPOSITION 1. *Let $L(F_n)$ be the free group factor on n generators with the tracial state τ , and u_1, \dots, u_n be the standard generators of $L(F_n)$. For each $m, k \geq 1$, let*

$$\Omega_m(k) = \{(U_1, \dots, U_n) \in \mathcal{U}(k)^n \mid |\tau_k(U_{i_1}^{\epsilon_1} \cdots U_{i_p}^{\epsilon_p}) - \tau(u_{i_1}^{\epsilon_1} \cdots u_{i_p}^{\epsilon_p})| < \frac{1}{m} \\ \text{for all } 1 \leq p \leq m, 1 \leq i_1, \dots, i_p \leq n, \{\epsilon_1, \dots, \epsilon_p\} \subset \{1, *\}\}.$$

Then

$$\lim_{k \rightarrow \infty} \mu_k(\Omega_m(k)) = 1,$$

where μ_k is normalized Haar measure on the compact group $\mathcal{U}(k)^n$.

THEOREM 3. *Suppose $L(F_n)$ is the free group factor on n generators with $n \geq 2$. Then Voiculescu's embedding dimension $\delta_0^\omega(L(F_n)) \geq 2$ and $\mathcal{H}_1^\omega(L(F_n)) = \infty$.*

PROOF. It follows from Proposition 1 that, for every $m \geq 1$, there are some positive integer k_m and a sequence of subsets $\{\Omega_m(k)\}_{k=k_m}^\infty$ such that

$$\mu_k(\Omega_m(k)) \geq \frac{1}{2}, \quad \text{for } k \geq k_m,$$

where μ_k is normalized Haar measure on the compact group $\mathcal{U}(k)^n$.

Let $\xi = \{k_m\}_{m=1}^\infty$. It is easy to see that $\xi \in \Xi$. For each $R > 1$ and such ξ , consider the sequence $\{\Sigma_k\}_{k=1}^\infty$ such that

$$\Sigma_k = \Omega_m(k), \quad \text{when } k_m \leq k < k_{m+1}.$$

It is not hard to verify that

$$\prod_{k=1}^\infty \Sigma_k \subset \mathfrak{N}_{R,\xi}(u_1, \dots, u_n).$$

So,

$$\Omega_m(k) = \Sigma_k \subset \mathbb{P}_k(\mathfrak{N}_{R,\xi}(u_1, \dots, u_n)) \quad \text{when } k_m \leq k < k_{m+1}.$$

Hence

$$\mu_k(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(u_1, \dots, u_n))) \geq \frac{1}{2} \quad \text{for all } k \geq 1.$$

Note there exists constant c (not depending on k) such that

$$\mu_k(\text{Ball}((U_1, \dots, U_n), \delta)) \leq (c\delta)^{nk^2}, \quad \forall 0 < \delta < 1,$$

where $\text{Ball}((U_1, \dots, U_n), \delta)$ is a ball centered at (U_1, \dots, U_n) with radius δ (with respect to 2-norm) in $(\mathcal{U}(k))^n$. A standard argument on covering numbers shows that

$$\delta_0^\omega(u_1, \dots, u_n) = n.$$

Similarly, there exist a constants C (not depending on k) such that

$$\mu_k(\mathcal{U}((U_1, \dots, U_n), \delta)) \leq (C\delta)^{(n-1)k^2}, \quad \forall 0 < \delta < 1,$$

where $\mathcal{U}((U_1, \dots, U_n), \delta)$ is a unitary orbit centered at (U_1, \dots, U_n) with radius δ (with respect to 2-norm) in $(\mathcal{U}(k))^n$. A standard arguments on unitary orbit covering number shows

$$\mathcal{H}_1^\omega(u_1, \dots, u_n) = \infty.$$

Thus, from Theorem 1, we have $\delta_0^\omega(L(F_n)) \geq 2$ and $\mathcal{H}_1^\omega(L(F_n)) = \infty$. \square

REMARK 9. *It seems that \mathcal{H}_1^ω does not provide us with more insights into the isomorphism problem of free group factors because $\mathcal{H}_1^\omega(L(F_n)) = \infty$ for all $n \geq 2$. But it will provide us with useful information when von Neumann algebra is not free group factors, which we will see in next sections.*

5. Some lemmas on covering number and orbit covering number

In this section, we are going to compute the covering numbers and orbit-covering numbers of some sets. We start with a definition, which is just for our convenience.

DEFINITION 9. *A unitary matrix U in $\mathcal{M}_k(\mathbb{C})$ is a Haar unitary matrix if $\tau_k(U^m) = 0$ for all $1 \leq m < k$ and $\tau_k(U^k) = 1$.*

We have the following lemma.

LEMMA 3. *Let V_1 be a Haar unitary matrix and V_2 be a unitary matrix in $\mathcal{M}_k(\mathbb{C})$. For every $\delta > 0$, let*

$$\Omega(V_1, V_2; \delta) = \{U \in \mathcal{U}(k) \mid \|UV_1 - V_2U\|_2 \leq \delta\}.$$

Then, for every $r > \delta$, there exists a set $\{\text{Ball}(U_\lambda; \frac{4\delta}{r})\}_{\lambda \in \Lambda}$ of $\frac{4\delta}{r}$ -balls in $\mathcal{U}(k)$ that cover $\Omega(V_1, V_2; \delta)$ with the cardinality of Λ satisfying $|\Lambda| \leq \left(\frac{3r}{2\delta}\right)^{4rk^2}$.

SKETCH OF PROOF. Let D be a diagonal unitary matrix, $\text{diag}(\lambda_1, \dots, \lambda_k)$, where λ_j is the j -th root of unity 1. Since V_1 is a Haar unitary matrices, there exists W_1 in $\mathcal{U}(k)$ such that $V_1 = W_1DW_1^*$. Assume that μ_1, \dots, μ_k are the eigenvalues of V_2 . Then there is some unitary matrix W_2 such that $V_2 = W_2D_2W_2^*$, where $D_2 = \text{diag}(\mu_1, \dots, \mu_k)$. Let $\tilde{\Omega}(\delta) = \{U \in \mathcal{U}(k) \mid \|UD - D_2U\|_2 \leq \delta\}$. Clearly

$\Omega(V_1, V_2; \delta) = \{W_2^* U W_1 \mid U \in \tilde{\Omega}(\delta)\}$; whence $\tilde{\Omega}(\delta)$ and $\Omega(V_1, V_2; \delta)$ have the same covering numbers.

Let $\{e_{st}\}_{s,t=1}^k$ be the canonical system of matrix units of $\mathcal{M}_k(\mathbb{C})$. Let

$$\mathcal{S}_1 = \text{span}\{e_{st} \mid |\lambda_s - \mu_t| < r\} \quad \mathcal{S}_2 = M_k(\mathbb{C}) \ominus \mathcal{S}_1.$$

For every $U = \sum_{s,t=1}^k x_{st} e_{st}$ in $\tilde{\Omega}(\delta)$, with $x_{st} \in \mathbb{C}$, let $T_1 = \sum_{e_{st} \in \mathcal{S}_1} x_{st} e_{st} \in \mathcal{S}_1$ and $T_2 = \sum_{e_{st} \in \mathcal{S}_2} x_{st} e_{st} \in \mathcal{S}_2$. But

$$\begin{aligned} \delta^2 &\geq \|UD - D_2 U\|_2^2 = \sum_{s,t=1}^k |(\lambda_s - \mu_t)x_{st}|^2 \geq \sum_{e_{st} \in \mathcal{S}_2} |(\lambda_s - \mu_t)x_{st}|^2 \\ &\geq r^2 \sum_{e_{st} \in \mathcal{S}_2} |x_{st}|^2 = r^2 \|T_2\|_2^2. \end{aligned}$$

Hence $\|T_2\|_2 \leq \frac{\delta}{r}$. Note that $\|T_1\|_2 \leq \|U\|_2 = 1$ and $\dim_{\mathbb{R}} \mathcal{S}_1 \leq 4rk^2$ (see [8]). By standard arguments on covering numbers, we know that $\tilde{\Omega}(\delta)$ can be covered by a set $\{Ball(A^\lambda; \frac{2\delta}{r})\}_{\lambda \in \Lambda}$ of $\frac{2\delta}{r}$ -balls in $\mathcal{M}_k(\mathbb{C})$ with $|\Lambda| \leq \left(\frac{3r}{2\delta}\right)^{4rk^2}$. Because $\tilde{\Omega}(\delta) \subset \mathcal{U}(k)$, after replacing A^λ by a unitary U^λ in $Ball(A^\lambda, \frac{2\delta}{r})$, we obtain a set $\{Ball(U^\lambda; \frac{4\delta}{r})\}_{\lambda \in \Lambda}$ of $\frac{4\delta}{r}$ -balls in $\mathcal{U}(k)$ that cover $\tilde{\Omega}(\delta)$ with the cardinality of Λ satisfying $|\Lambda| \leq \left(\frac{3r}{2\delta}\right)^{4rk^2}$. Therefore the same result holds for $\Omega(V_1, V_2; \delta)$. \square

With the notations as above, we have following lemmas.

LEMMA 4. Suppose $R > 1$, $0 < r, \delta < 1$. Let $r_1 = \frac{r\delta}{128R}$. Suppose Γ is a subset of

$$(\mathcal{M}_k(\mathbb{C})^n \times \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_k(\mathbb{C}))_R$$

such that every

$$(A_1, \dots, A_n, U, V, W) \in \Gamma \subset (\mathcal{M}_k(\mathbb{C})^n \times \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_k(\mathbb{C}))_R$$

satisfies

- (i) There exist some Haar unitary matrix V_1 and unitary matrices U_1, W_1 in $\mathcal{U}(k)$ such that $\|V - V_1\|_2 < r_1$, $\|U - U_1\|_2 < r_1$ and $\|W - W_1\|_2 < r_1$;
- (ii) $\|UV - WU\|_2 < r_1$.

Let

$$\begin{aligned} \Gamma_1 &= \{(A_1, \dots, A_n, U) \in (\mathcal{M}_k(\mathbb{C})^n \times \mathcal{M}_k(\mathbb{C})) \mid \\ &\quad \exists V, W \text{ such that } (A_1, \dots, A_n, U, V, W) \in \Gamma\} \\ \Gamma_2 &= \{(A_1, \dots, A_n, V, W) \in (\mathcal{M}_k(\mathbb{C})^n \times \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_k(\mathbb{C})) \mid \\ &\quad \exists U, \text{ such that } (A_1, \dots, A_n, U, V, W) \in \Gamma\}. \end{aligned}$$

Then we have

$$\nu(\Gamma_1, \delta) \leq \nu(\Gamma_2, \frac{r\delta}{128}) \cdot \left(\frac{24}{\delta}\right)^{4rk^2},$$

where $\nu(\Gamma_1, \delta)$, or $\nu(\Gamma_2, \frac{r\delta}{128})$, is the unitary orbit covering number of the set Γ_1 , or Γ_2 respectively, with radius δ_1 , or $\frac{r\delta}{128}$ respectively.

PROOF. By the definition of unitary orbit covering number, we know that Γ_2 can be covered by a collection of $\frac{r\delta}{128}$ -orbit-balls $\{\mathcal{U}(A_1^\lambda, \dots, A_n^\lambda, V^\lambda, W^\lambda; \frac{r\delta}{128})\}_{\lambda \in \Lambda}$ such that the cardinality of Λ satisfies $|\Lambda| = \nu(\Gamma_2, \frac{r\delta}{128})$.

For every $(A_1, \dots, A_n, U, V, W) \in \Gamma$, we know

- (i) (A_1, \dots, A_n, V, W) is contained in Γ_2 ;
- (ii) There exist some Haar unitary matrix V_1 and unitary matrices U_1, W_1 in $\mathcal{U}(k)$ such that $\|V - V_1\|_2 < r_1$, $\|U - U_1\|_2 < r_1$ and $\|W - W_1\|_2 < r_1$;
- (iii) $\|UV - WU\|_2 < r_1$.

From (i), it follows that there are some λ in Λ such that

$$(A_1, \dots, A_n, V, W) \in \mathcal{U}(A_1^\lambda, \dots, A_n^\lambda, V^\lambda, W^\lambda; \frac{r\delta}{128}).$$

Or, there is some unitary matrix X in $\mathcal{U}(k)$ such that

$$\|(A_1, \dots, A_n, V, W) - X(A_1^\lambda, \dots, A_n^\lambda, V^\lambda, W^\lambda)X^*\| \leq \frac{r\delta}{128}.$$

Combining with (ii) and (iii), we obtain that

$$\|U_1 X V^\lambda X^* - X W^\lambda X^* U_1\|_2 < r_1 + 2\frac{r\delta}{128} + 2Rr_1,$$

and

$$\|V_1 - X V^\lambda X^*\|_2 \leq r_1 + \frac{r\delta}{128}, \quad \|W_1 - X W^\lambda X^*\|_2 \leq r_1 + \frac{r\delta}{128}.$$

It follows that there are some Haar unitary matrix \tilde{V}^λ and unitary matrix \tilde{W}^λ in $\mathcal{U}(k)$ such that

$$\|\tilde{V}^\lambda - V^\lambda\|_2 \leq r_1 + \frac{r\delta}{128}, \quad \|\tilde{W}^\lambda - W^\lambda\|_2 \leq r_1 + \frac{r\delta}{128}.$$

Replace this V^λ by such Haar unitary matrix \tilde{V}^λ and this W^λ by such unitary matrix \tilde{W}^λ , when V^λ is not a Haar unitary matrix and W^λ is not a unitary matrix. Therefore, we have

$$\|U_1 X \tilde{V}^\lambda X^* - X \tilde{W}^\lambda X^* U_1\|_2 < 2(r_1 + \frac{r\delta}{128}) + (r_1 + 2\frac{r\delta}{128} + 2Rr_1) \leq \frac{r\delta}{16}.$$

On the other hand, it follows from Lemma 2, there exists a set $\{Ball(U^{\lambda\sigma}; \frac{\delta}{4})\}_{\sigma \in \Sigma}$ of $\frac{\delta}{4}$ -balls that covers $\Omega(\tilde{V}^\lambda, \tilde{W}^\lambda; \frac{r\delta}{16})$ such that $|\Sigma| \leq (\frac{24}{\delta})^{4rk^2}$. Thus there is some σ in Σ such that

$$\|X^* U_1 X - U^{\lambda\sigma}\| \leq \frac{\delta}{4}.$$

It induces that

$$\|(A_1, \dots, A_n, U, V, W) - X(A_1^\lambda, \dots, A_n^\lambda, U^{\lambda\sigma}, V^\lambda, W^\lambda)X^*\|_2 \leq \frac{r\delta}{128} + r_1 + \frac{\delta}{4} \leq \frac{\delta}{2}.$$

From the definition of unitary orbit covering number it follows that

$$(A_1, \dots, A_n, U, V, W) \in U(A_1^\lambda, \dots, A_n^\lambda, U^{\lambda\sigma}, V^\lambda, W^\lambda; \delta)$$

Hence,

$$\nu(\Gamma_1, \delta) \leq \nu(\Gamma, \delta) \leq |\Lambda||\Sigma| \leq \nu(\Gamma_2, \frac{r\delta}{128}) \cdot \left(\frac{24}{\delta}\right)^{4rk^2}.$$

□

LEMMA 5. Suppose $R > 1$, $0 < r, \delta < 1$. Let $r_1 = \frac{r\delta}{96R}$. Suppose Γ is a subset of

$$(\mathcal{M}_k(\mathbb{C})^n \times (\mathcal{M}_k(\mathbb{C})^p \times \mathcal{M}_k(\mathbb{C}))_R$$

such that every

$$(A_1, \dots, A_n, C_1, \dots, C_p, U) \in \Gamma \subset (\mathcal{M}_k(\mathbb{C})^n \times \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_k(\mathbb{C}) \times \mathcal{M}_k(\mathbb{C}))_R$$

satisfies that there exists some Haar unitary matrix U_1 in $\mathcal{U}(k)$ such that

$$\|U - U_1\|_2 < r_1.$$

Let

$$\begin{aligned} \Gamma_1 &= \{(A_1, \dots, A_n, U) \in (\mathcal{M}_k(\mathbb{C})^n \times \mathcal{M}_k(\mathbb{C})) \mid \\ &\quad \exists C_1, \dots, C_p \text{ such that } (A_1, \dots, A_n, C_1, \dots, C_p, U) \in \Gamma\} \\ \Gamma_2 &= \{(C_1, \dots, C_p, U) \in (\mathcal{M}_k(\mathbb{C})^p \times \mathcal{M}_k(\mathbb{C})) \mid \\ &\quad \exists A_1, \dots, A_n \text{ such that } (A_1, \dots, A_n, C_1, \dots, C_p, U) \in \Gamma\}. \end{aligned}$$

Then we have

$$\nu(\Gamma, 2n\delta) \leq \nu(\Gamma_1, \frac{r\delta}{96}) \cdot \nu(\Gamma_2, \frac{r\delta}{96}) \cdot \left(\frac{18R}{\delta}\right)^{4rk^2}$$

where $\nu(\Gamma, 2n\delta)$, or $\nu(\Gamma_i, \frac{r\delta}{96})$ ($i = 1, 2$), is the unitary orbit covering number of the set Γ , or Γ_i ($i = 1, 2$) respectively, with radius $2n\delta_1$, or $\frac{r\delta}{96}$ respectively.

PROOF. By the definition of unitary orbit covering numbers of the sets Γ_1 and Γ_2 , there exists a set $\{\mathcal{U}(B_1^\lambda, \dots, B_n^\lambda, U_\lambda; \frac{r\delta}{96R})\}_{\lambda \in \Lambda}$ of $\frac{r\delta}{96R}$ -orbit-balls in $(\mathcal{M}_k(\mathbb{C})^{n+1})_R$ covering Γ_1 with $|\Lambda| = \nu(\Gamma_1, \frac{r\delta}{96R})$. Also there exists a set $\{\mathcal{U}(D_1^\sigma, \dots, D_p^\sigma, U_\sigma; \frac{r\delta}{96R})\}_{\sigma \in \Sigma}$ of $\frac{r\delta}{96R}$ -orbit-balls in $(\mathcal{M}_k(\mathbb{C})^{p+1})_R$ that cover Γ_2 with $|\Sigma| = \nu(\Gamma_2, \frac{r\delta}{96R})$.

For each $(A_1, \dots, A_n, C_1, \dots, C_p, U)$ in Γ , we know the following hold.

- (i) (A_1, \dots, A_n, U) is contained in Γ_1 ;
- (ii) (C_1, \dots, C_p, U) is contained in Γ_2 ;
- (iii) There exists some Haar unitary matrix U_1 in $\mathcal{U}(k)$ such that

$$\|U - U_1\|_2 \leq r_1.$$

From (i) and (ii), there exist some λ in Λ , σ in Σ such that

$$(A_1, \dots, A_n, U) \in U(B_1^\lambda, \dots, B_n^\lambda, U_\lambda; \frac{r\delta}{96R})$$

$$(C_1, \dots, C_p, U) \in U(D_1^\sigma, \dots, D_p^\sigma, U_\sigma; \frac{r\delta}{96R})$$

i.e., there exist unitary matrices W_1, W_2 in $\mathcal{U}(k)$ such that

$$\|(A_1, \dots, A_n, U) - (W_1 B_1^\lambda W_1^*, \dots, W_1 B_n^\lambda W_1^*, W_1 U_\lambda W_1^*)\|_2 \leq \frac{r\delta}{96R}$$

$$\|(C_1, \dots, C_p, U) - (W_2 D_1^\sigma W_2^*, \dots, W_2 D_p^\sigma W_2^*, W_2 U_\sigma W_2^*)\|_2 \leq \frac{r\delta}{96R}.$$

Combining with (iii), we have that

$$\|U_1 - W_1 U_\lambda W_1^*\|_2 \leq r_1 + \frac{r\delta}{96R} \leq \frac{r\delta}{48R}, \quad \|U_1 - W_2 U_\sigma W_2^*\|_2 \leq r_1 + \frac{r\delta}{96R} \leq \frac{r\delta}{48R}.$$

Therefore there are two Haar unitary matrices V_λ, V_σ such that

$$\|U_\lambda - V_\lambda\|_2 \leq \frac{r\delta}{48R}, \quad \|U_\sigma - V_\sigma\|_2 \leq \frac{r\delta}{48R}.$$

Replace these U_λ, U_σ by Haar unitary matrices V_λ, V_σ when U_λ, U_σ are not Haar unitary matrices. It follows

$$\|W_2^* W_1 V_\lambda - V_\sigma W_2^* W_1\|_2 = \|W_1 V_\lambda W_1^* - W_2 V_\sigma W_2^*\|_2 \leq \frac{r\delta}{12R}.$$

Since V_λ, V_σ are Haar unitary matrices in $\mathcal{M}_k(\mathbb{C})$, by Lemma 3 we know that there exists a set $\{Ball(U_{\lambda\sigma\gamma}; \frac{\delta}{3R})\}_{\gamma \in \mathcal{I}_k}$ of $\frac{\delta}{3R}$ -balls in $\mathcal{U}(k)$ that cover $\Omega(V_\lambda, V_\sigma; \frac{r\delta}{12R})$ with the cardinality of \mathcal{I}_k never exceeding $(\frac{18R}{\delta})^{4rk^2}$. Then there exists some $\gamma \in \mathcal{I}_k$ such that $\|W_2^* W_1 - U_{\lambda\sigma\gamma}\|_2 \leq \frac{\delta}{3R}$. This in turn implies

$$\|(A_1, \dots, A_n, C_1, \dots, C_p, U) - (W_2 U_{\lambda\sigma\gamma} B_1^\lambda U_{\lambda\sigma\gamma}^* W_2^*, \dots, W_2 U_{\lambda\sigma\gamma} B_n^\lambda U_{\lambda\sigma\gamma}^* W_2^*, \\ W_2 D_1^\sigma W_2^*, \dots, W_2 D_p^\sigma W_2^*, W_2 U_\sigma W_2^*)\|_2 \leq n\delta$$

for some $\lambda \in \Lambda_k, \sigma \in \Sigma_k, \gamma \in \mathcal{I}_k$ and $W_2 \in \mathcal{U}(k)$, i.e.,

$$(A_1, \dots, A_n, C_1, \dots, C_p, U) \in \mathcal{U}(U_{\lambda\sigma\gamma} B_1^\lambda U_{\lambda\sigma\gamma}^*, \dots, U_{\lambda\sigma\gamma} B_n^\lambda U_{\lambda\sigma\gamma}^*, D_1^\sigma, \dots, D_p^\sigma, U_\sigma; 2n\delta).$$

From the definition of unitary orbit covering number it follows that

$$\nu(\Gamma, \delta) \leq |\Lambda| \cdot |\Sigma| \cdot \left(\frac{18R}{\delta}\right)^{4rk^2} = \nu(\Gamma_1, \frac{r\delta}{96}) \cdot \nu(\Gamma_2, \frac{r\delta}{96}) \cdot \left(\frac{18R}{\delta}\right)^{4rk^2}$$

□

6. The computation of \mathcal{H}_1^ω for some finite von Neumann algebras

In this section, we are going to compute $\mathcal{H}_1^\omega(\mathcal{N})$ for a finite von Neumann algebra \mathcal{N} , including type II_1 factors with property Γ , with Cartan subalgebras and nonprime type II_1 factors.

6.1. Embedding dimension. The same strategy as in the proof of Theorem 1 can be used to prove the following theorem whose proof is skipped here.

THEOREM 4. *Suppose \mathcal{N} is a finitely generated von Neumann algebra with a tracial state τ and \mathcal{N} can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$. Suppose $\{\mathcal{N}_j\}_{j=1}^\infty$ is an increasing sequence of von Neumann subalgebras of \mathcal{N} such that $\mathcal{N} = \overline{\bigcup_{j=1}^\infty \mathcal{N}_j}^{SOT}$. Then, for each $s \geq 0$,*

$$0 \leq \mathcal{H}_s^\omega(\mathcal{N}) \leq \liminf_{j \rightarrow \infty} \mathcal{H}_s^\omega(\mathcal{N}_j).$$

DEFINITION 10. *Suppose that \mathcal{N} is a diffuse von Neumann algebra with a tracial state τ . Then a unitary element u in \mathcal{N} is called a Haar unitary if $\tau(u^m) = 0$ when $m \neq 0$.*

THEOREM 5. *Suppose \mathcal{N} is a diffuse finitely generated von Neumann algebra with a tracial state τ and \mathcal{N} can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$. Suppose \mathcal{N}_1 is a diffuse von Neumann subalgebra of \mathcal{N} and u is a unitary element in \mathcal{N} such that $\{\mathcal{N}_1, u\}$ generates \mathcal{M} as a von Neumann algebra. If there exist Haar unitary elements v_1, v_2, \dots and unitary elements w_1, w_2, \dots in \mathcal{N}_1 such that $\|v_n u - u w_n\|_2 \rightarrow 0$, then $\mathcal{H}_1^\omega(\mathcal{N}) \leq \mathcal{H}_1^\omega(\mathcal{N}_1)$. In particular, if there are Haar unitary elements v, w in \mathcal{N}_1 , such that $vu = uw$, then $\mathcal{H}_1^\omega(\mathcal{N}) \leq \mathcal{H}_1^\omega(\mathcal{N}_1)$.*

PROOF. Suppose that $\{x_1, \dots, x_n\}$ is a family of generators of \mathcal{N}_1 . Then we know that $\{x_1, \dots, x_n, u\}$ is a family of generators of \mathcal{M} .

For every $0 < \delta < 1$, $0 < r < 1$, there exist an integer $p > 0$ and a Haar unitary element v_p , and a unitary element w_p in \mathcal{N}_1 such that

$$\|v_p u - u w_p\|_2 < \frac{r\delta}{130}.$$

Note that $\{x_1, \dots, x_n, v_p, w_p\}$ is also a family of generators of \mathcal{N}_1 .

For $R > 1$, $\xi = \{k_m\}_{m=1}^\infty$ in Ξ , let $r_1 = \frac{r\delta}{128R}$. By the definition of

$$\mathfrak{N}_{R,\xi}(x_1, \dots, x_n, u, v_p, w_p)$$

and Lemma 2, there is some $m_0 \geq 0$ such that every

$$((A_{1,k})_k, \dots, (A_{n,k})_k, (U_k)_k, (V_k)_k, (W_k)_k) \in \mathfrak{N}_{R,\xi}(x_1, \dots, x_n, u, v_p, w_p),$$

satisfies, when $k \geq k_{m_0}$,

- (i) there exist some Haar unitary matrix $V_{1,k}$ and unitary matrices $U_{1,k}, W_{1,k}$ in $\mathcal{U}(k)$ such that

$$\|V_k - V_{1,k}\|_2 < r_1, \|U_k - U_{1,k}\|_2 < r_1 \text{ and } \|W - W_{1,k}\|_2 < r_1;$$

- (ii) $\|U_k V_k - W_k U_k\|_2 < r_1$.

Apply Lemma 3 by letting $\Gamma = \mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n, u, v_p, w_p))$. We get that

$$\begin{aligned} & \nu(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n, u : v_p, w_p)), \delta) \\ & \leq \nu(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n, v_p, w_p : u)), \frac{r\delta}{128}) \cdot \left(\frac{24}{\delta}\right)^{4rk^2} \\ & \leq \nu(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n, v_p, w_p)), \frac{r\delta}{128}) \cdot \left(\frac{24}{\delta}\right)^{4rk^2}. \end{aligned}$$

Hence, by the definition of embedding dimension, we have shown

$$\begin{aligned} 0 & \leq \mathcal{H}_1^\omega(x_1, \dots, x_n, u : v_p, w_p; R, \xi, \delta) \\ & \leq \lim_{k \rightarrow \omega} \frac{\log(\nu(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n, v_p, w_p)), \frac{r\delta}{128}) \cdot \left(\frac{24}{\delta}\right)^{4rk^2})}{k^2} \\ & = \lim_{k \rightarrow \omega} \frac{\log(\nu(\mathbb{P}_k(\mathfrak{N}_{R,\xi}(x_1, \dots, x_n, v_p, w_p)), \frac{r\delta}{128}))}{k^2} + 4r \cdot (\log 24 - \log \delta) \\ & \leq \mathcal{H}_s^\omega(x_1, \dots, x_n, v_p, w_p) + 4r \cdot (\log 24 - \log \delta) \\ & = \mathcal{H}_s^\omega(x_1, \dots, x_n) + 4r \cdot (\log 24 - \log \delta). \end{aligned}$$

Because r is an arbitrarily small positive number, we have

$$\mathcal{H}_1^\omega(x_1, \dots, x_n, u : v_p, w_p) \leq \mathcal{H}_s^\omega(x_1, \dots, x_n).$$

By Lemma 1 and Theorem 1, $\mathcal{H}_1^\omega(\mathcal{N}) \leq \mathcal{H}_1^\omega(\mathcal{N}_1)$. \square

THEOREM 6. *Suppose \mathcal{N} is a finitely generated von Neumann algebra with a tracial state τ and \mathcal{N} can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$. Suppose \mathcal{N} is generated by von Neumann subalgebras \mathcal{N}_1 and \mathcal{N}_2 of \mathcal{N} . If $\mathcal{N}_1 \cap \mathcal{N}_2$ is a diffuse von Neumann subalgebra of \mathcal{N} , then*

$$\mathcal{H}_1^\omega(\mathcal{N}) \leq \mathcal{H}_1^\omega(\mathcal{N}_1) + \mathcal{H}_1^\omega(\mathcal{N}_2).$$

PROOF. Suppose that $\{x_1, \dots, x_n\}$ is a family of generators of \mathcal{N}_1 and $\{y_1, \dots, y_p\}$ a family of generators of \mathcal{N}_2 . Since $\mathcal{N}_1 \cap \mathcal{N}_2$ is a diffuse von Neumann subalgebra, we can find a Haar unitary u in $\mathcal{N}_1 \cap \mathcal{N}_2$.

For every $R > 1 + \max_{1 \leq i \leq n, 1 \leq j \leq p} \{\|x_i\|, \|y_j\|\}$, $0 < \delta < \frac{1}{2n}$, $0 < r < 1$ and $\xi \in \Xi$, let $r_1 = \frac{r\delta}{96R}$. From the definition of $\mathcal{N}_{R,\xi}(x_1, \dots, x_n, y_1, \dots, y_p, u)$ and Lemma 2, there is some $m_0 \geq 0$ such that, every

$$((A_{1,k})_k, \dots, (A_{n,k})_k, (B_{1,k})_k, \dots, (B_{p,k})_k, (U_k)_k) \in \mathcal{N}_{R,\xi}(x_1, \dots, x_n, y_1, \dots, y_p, u)$$

satisfies that, when $k \geq k_{m_0}$, there exists a Haar unitary matrix $U_{1,k}$ such that

$$\|U_k - U_{1,k}\|_2 \leq r_1,$$

Apply Lemma 5 by letting $\Gamma = \mathbb{P}_k(\mathcal{N}_{R,\xi}(x_1, \dots, x_n, y_1, \dots, y_p, u))$. We get that

$$\begin{aligned} \nu(\mathbb{P}_k(\mathcal{N}_{R,\xi}(x_1, \dots, x_n, y_1, \dots, y_p, u)), 2n\delta) &\leq \nu(\mathbb{P}_k(\mathcal{N}_{R,\xi}(x_1, \dots, x_n, u : y_1, \dots, y_p)), \frac{r\delta}{96}) \\ &\quad \cdot \nu(\mathbb{P}_k(\mathcal{N}_{R,\xi}(y_1, \dots, y_p, u : x_1, \dots, x_n)), \frac{r\delta}{96}) \cdot \left(\frac{18R}{\delta}\right)^{4rk^2} \\ &\leq \nu(\mathbb{P}_k(\mathcal{N}_{R,\xi}(x_1, \dots, x_n, u)), \frac{r\delta}{96}) \cdot \nu(\mathbb{P}_k(\mathcal{N}_{R,\xi}(y_1, \dots, y_p, u)), \frac{r\delta}{96}) \cdot \left(\frac{18R}{\delta}\right)^{4rk^2} \end{aligned}$$

Now it not hard to show that

$$\begin{aligned} \mathcal{H}_1^\omega(x_1, \dots, x_n, y_1, \dots, y_p, u; R, \xi, 2n\delta) \\ \leq \mathcal{H}_1^\omega(x_1, \dots, x_n, u) + \mathcal{H}_1^\omega(y_1, \dots, y_p, u) + 4r \cdot (\log 18R - \log \delta). \end{aligned}$$

Because r is an arbitrarily small positive number, we have

$$\mathcal{H}_1^\omega(x_1, \dots, x_n, y_1, \dots, y_p, u) \leq \mathcal{H}_1^\omega(x_1, \dots, x_n, u) + \mathcal{H}_1^\omega(y_1, \dots, y_p, u).$$

By Theorem 1, we obtain,

$$\mathcal{H}_1^\omega(\mathcal{N}) \leq \mathcal{H}_1^\omega(\mathcal{N}_1) + \mathcal{H}_1^\omega(\mathcal{N}_2).$$

□

Now we are able to going to compute values of \mathcal{H}_1^ω for many specific type II_1 factors.

THEOREM 7. *Suppose that \mathcal{N} is a type II_1 factor with Cartan subalgebras and \mathcal{N} can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$. Then $\mathcal{H}_1^\omega(\mathcal{N}) = 0$.*

PROOF. It follows from [15] that \mathcal{N} is generated by two self-adjoint elements of \mathcal{N} (see also [12]). Note there is a maximal abelian von Neumann subalgebra \mathcal{A} of \mathcal{N} such that $N(\mathcal{A})$, the normalizers of \mathcal{A} in \mathcal{N} , generates \mathcal{N} where $N(\mathcal{A})$ is the group of all these unitary elements u in \mathcal{N} such that $u^* \mathcal{A} u = \mathcal{A}$. Now it follows directly from Theorem 1, Theorem 4 and Theorem 5 that $\mathcal{H}_1^\omega(\mathcal{N}) = 0$. □

COROLLARY 1. *Suppose that \mathcal{R} is the hyperfinite type II_1 factor. Then $\mathcal{H}_1^\omega(\mathcal{R}) = 0$.*

THEOREM 8. *Suppose that \mathcal{N} is type II_1 factor with property Γ and \mathcal{N} can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$. Then $\mathcal{H}_1^\omega(\mathcal{N}) = 0$.*

PROOF. It follows from [6] that \mathcal{N} is generated by two self-adjoint elements (see also [12]). From Theorem 5.3 of [1], it follows that there is a hyepfinite II_1 subfactor \mathcal{R} of \mathcal{N} such that $\mathcal{N}' \cap \mathcal{R}^\omega$ is diffuse. Now it follows from Corollary 1, Theorem 4 and Theorem 5 that $\mathcal{H}_1^\omega(\mathcal{N}) = 0$. □

The following two theorems also follows directly from Theorem 1, Theorem 4 and Theorem 5, whose proofs are skipped here.

THEOREM 9. *Suppose that \mathcal{N} is a non-prime type II_1 factor, i.e. the tensor product of two type II_1 subfactors, and \mathcal{N} can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$. Then $\mathcal{H}_1^\omega(\mathcal{N}) = 0$.*

THEOREM 10. *Suppose that $SL(2n+1, \mathbb{Z})$ is the special linear group with the integer entries. Then $\mathcal{H}_1^\omega(L(SL(2n+1, \mathbb{Z}))) = 0$.*

REMARK 10. *We know that $\mathcal{H}_s^\omega(\mathcal{N})$ is a decreasing function of $s \geq 0$. The following question is of interest to us. Suppose that \mathcal{N} is a type II_1 factor with $\mathcal{H}_1^\omega(\mathcal{N}) = 0$. Can we find some number t such that $\mathcal{H}_t^\omega(\mathcal{N}) > 0$? How about $L(F_2) \otimes L(F_2)$? How about type II_1 factors with Cartan subalgebras?*

6.2. Voiculescu's embedding dimension of thin factors. The concept of thin factor was introduced by S. Popa (also see [6]). It was known in [6] that free group factors on n generators with $n \geq 4$ are not thin factors. This concept was further generalized to \mathfrak{K} -thin in [10]. It was shown there that free group factors on n generators with $n \geq 4$ are not \mathfrak{K} -thin factors. Here, we are going to consider Voiculescu's embedding dimensions of these "thin" factors.

THEOREM 11. *Suppose that \mathcal{N} is a finitely generated type II_1 factor with a tracial state τ and \mathcal{N} can be faithfully trace-preserving embedded into $\mathcal{M}_k(\mathbb{C})^\omega$. Suppose there exist two subalgebras $\mathcal{N}_0, \mathcal{N}_1$ and n -vectors ξ_1, \dots, ξ_n in $L^2(\mathcal{M}, \tau)$ such that $0 \leq \mathcal{H}_1^\omega(\mathcal{N}_0), \mathcal{H}_1^\omega(\mathcal{N}_1) < \infty$, and $\overline{\text{span}}^{\|\cdot\|_2} \mathcal{N}_0 \{\xi_1, \dots, \xi_n\} \mathcal{N}_1 = L^2(\mathcal{M}, \tau)$. Then Voiculescu's embedding dimension of \mathcal{N} satisfies*

$$\delta_0^\omega(\mathcal{N}) \leq 1 + 2n$$

PROOF. The proof of this theorem is just a slight modification of the proof of Theorem 7 in [10]. □

The following corollary follows easily from the preceding theorem.

COROLLARY 2. *Suppose \mathcal{N} is a type II_1 factor with a simple masa or is a thin factor and \mathcal{N} can be faithfully embedded into $\mathcal{M}_k(\mathbb{C})^\omega$. Then Voiculescu's embedding dimension $\delta_0(\mathcal{N}) \leq 3$.*

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